Physics 281

Lecture III → More on Turbulence / Fractals and Intermittency

→ Prequel to Turbulence : Richardson

- first 'indicator' of turbulent cascade in 3D was Richardson observation of super-diffusive separation of balloons in boundary layer

$$\Rightarrow \quad \underbrace{\langle \delta \ell^2 \rangle}_{\text{separation}} \sim t^3 \qquad (\text{not } \cancel{\text{diffusion}} \ \Delta t \sim \langle \delta\ell^2 \rangle)$$

ultimately: $\langle \delta \ell^2 \rangle \sim \epsilon t^3$

and not exponential divergence, aka Lyapunov exp., in chaos

⇒ suggests power law scaling of velocity (structure functions)

$$\delta V \sim \ell^{\alpha}$$

so

$$\frac{d\ell}{dt} \sim \ell^{\alpha}$$

⇒ comparable scales most effective in separatory

$\ell' \sim \ell$

$\ell' \ll \ell$ $\ell' \gg \ell$

$$\Rightarrow \frac{dl}{l^{\alpha}} = dt$$

$$l^{-\alpha+1} = t$$

$$\therefore \quad l \sim t^{1/(1-\alpha)}$$

$$\langle l^2 \rangle \sim t^3 \sim t^{2/(1-\alpha)}$$

$$\alpha = 1/3 .$$

$$\Rightarrow \boxed{\delta v(l) \sim l^{1/3}} \Rightarrow \langle \delta l^2 \rangle \sim t^3$$

↳ K41 scaling of $p=1$ structure function.

- point is that larger eddys have more energy, so rate of separation accelerates with scale (increase)
  $\Rightarrow$ <u>super</u>-diffusion of separation

- $$\frac{\delta v(l)^3}{l} \simeq \epsilon \Rightarrow \delta v(l) \sim \epsilon^{1/3} l^{1/3}$$

$\Rightarrow \langle \delta l^2 \rangle \sim \epsilon$ [redacted] t^3

$\Rightarrow$ Relative separation is excellent diagnostic of flow field dynamics.

N. B.

- separation on $\ell > \ell_0 \Rightarrow$ not super-diffusive
i.e. large eddy structure matters

$\ell < \ell_d \Rightarrow$ ~ strain, smooth

i.e. $\frac{d}{dt}\delta\ell \sim \nabla v\, \delta\ell$

$\Rightarrow$ exponential, as straining field smooth on that scale.

$\rightarrow$ $\delta\ell$ 'along' strain axis.

- why not exponential, a la Lyapunov exponent?

turbulence is spatially "rough", i.e.

$\delta v(\ell) \sim \epsilon^{1/3}\ell^{1/3}$

$$\lim_{\ell\to 0} \frac{v(x+\ell)-v(x)}{\ell} = \lim_{\ell\to 0} \frac{\delta v(\ell)}{\ell}$$

$\Rightarrow$ strain field not "smooth"

$$= \epsilon^{1/3}/\ell^{2/3}$$

$\Rightarrow$ develops rougher structure on smaller scales

strain increases on smaller scales.

→ Some Observations:

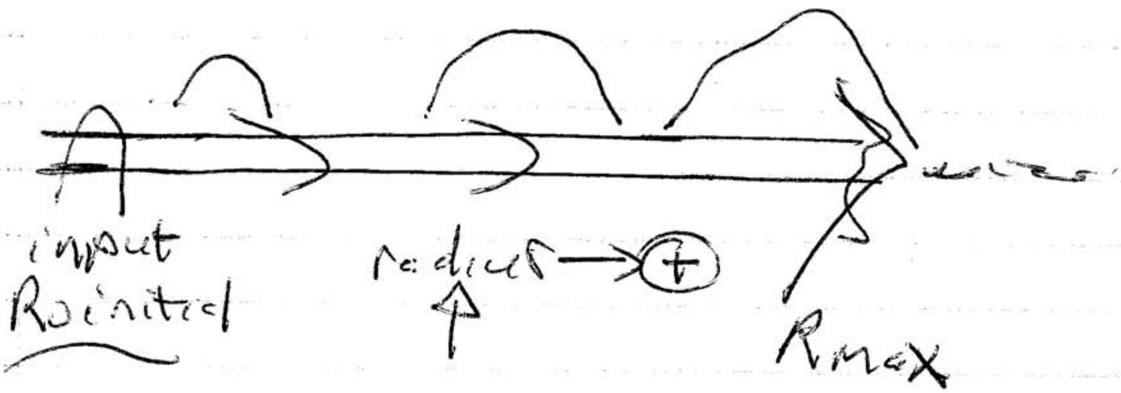
Can construct comparison of self-similarity phenomena:

	Blast Wave	Cascade
order param.	$R(t)$	$\delta V(\ell)$
outer scale	R_0 (initial)	ℓ_0
inner cutoff	R_{max}	ℓ_d
balance condition	$E \sim \rho R^3 \frac{R^2}{t^2}$	$\epsilon \sim \frac{\delta V(\ell)^3}{\ell}$
power law scaling	$R(t) \sim t^{2/5}$	$\delta V(\ell) \sim \epsilon^{1/3} \ell^{1/3}$
cut-off	$\rho \frac{R^2}{t^2} \sim P_{amb}$	$\frac{V(\ell)}{\ell} \sim \frac{\nu}{\ell^2} \rightarrow \ell_d$
	⇒ Exact self-similar solution ↑ i.e. solve for spatial str. of $\rho_0(t)\,\Theta(r/R(t))$	phenomenology (but successful)

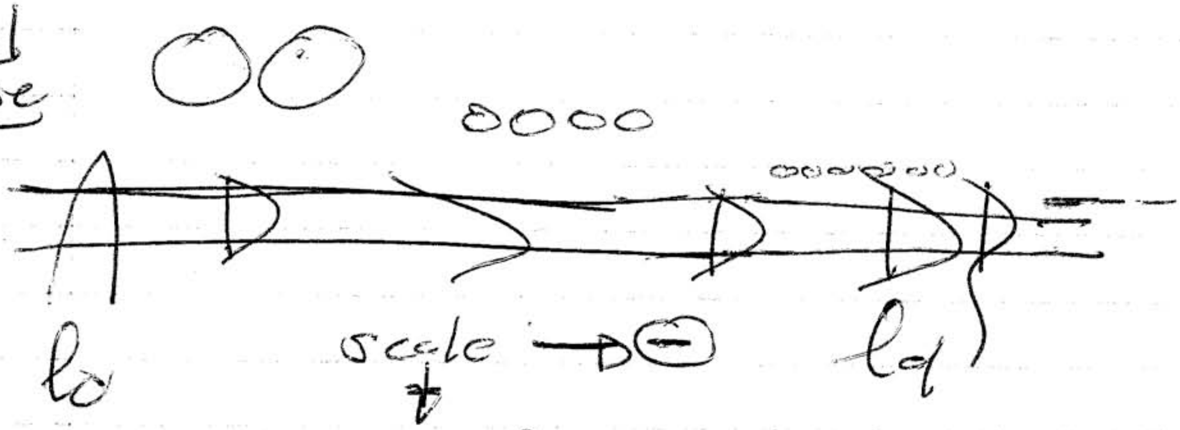
- both phenomena can viewed as 'flows' in scale space!

i.e.

blast



inertial range



n.b. $\epsilon \uparrow \Rightarrow l_d$ decreases, but I.R. same

also: no explicit dependence on R_0, l_0 except via E, ϵ!

n.b. observe with intermittency!

– flows

blast → $\frac{d}{dt}\left(R^3 V^2\right) = \frac{d}{dt}\left(R^5/t^2\right) = 0$

$\dot{R}/R \sim 2/5t$ ✓

turbulence →

and (I = intensity)

damping

$$\frac{\partial}{\partial t}(kI) = \frac{-\partial}{\partial \ln k}\left(k\sqrt{kI}\, kI\right) - \nu k^3 I$$

(see Kulsrud) flow

$$0 = \frac{-\partial}{\partial \ln k}\left(k\sqrt{kI}\, kI\right) - \nu k^3 I$$

i.e. $k_0 I(k_0) = V_0^2$

result ⇒

$$I = \frac{V_0^2 k_0^{2/3}}{k^{5/3}}\left[1 - \left(\frac{k}{k_0}\right)^{4/3}\frac{4}{4Re}\right]$$

K41

cut-off

→ Dissipation and Dissipative Structures !?

~ K41 phenomenology suggests uniform distribution of dissipation

l_d

i.e. dissipation on scale l_d, filling space

~ in reality;

→ distribution of dissipation is variable in intensity

→ not space filling / patchy (intermittent)

i.e.

→ some departure from K41 spectrum concomitant.

⇒ How characterize ?? ⇒ need a phenomenology, first.

⇒ Fractal Intermittency Models
(β - model) ....

Ref: ① Frisch, Sulem, Nelkin
② Frisch, "Turbulence + The Legacy of A.N. Kolmogorov"

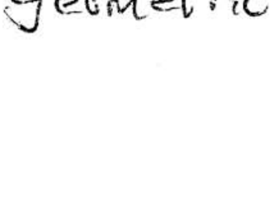
- Fractal ? why ?
- what does "dimension" mean ?

n.b. fractal concepts enable geometric phenomenology

→ Dimension

How define dimension:

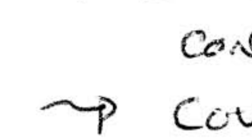
→ consider structure



[embedded in Cartesian space]

consider

→ covering: N-dimensional cubes (cubes - Cartesian) of size ϵ
(covering set by space structure is embedded in)



→ if $\tilde{N}(\epsilon)$ = # cubes to cover

$$\Rightarrow \boxed{D_0 = \lim_{\epsilon \to 0} \frac{\ln \tilde{N}(\epsilon)}{\ln (1/\epsilon)}}$$

Box-Counting Dimension:

$$\boxed{D_0 = \lim_{\epsilon \to 0} \frac{\ln \tilde{N}(\epsilon)}{\ln(1/\epsilon)}} \rightarrow \text{general definition}$$

check:

(a) Finite # points

p pts

$$D_0 = \lim_{\epsilon \to 0} \frac{\ln p}{\ln(1/\epsilon)}$$

$$= 0 \quad ✓$$

(b) Line = length ℓ

$$D_0 = \lim_{\epsilon \to 0} \frac{\ln \tilde{N}(\epsilon)}{\ln(1/\epsilon)}$$

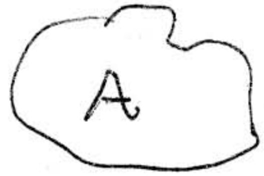
$\tilde{N} = \ell/\epsilon$

$$= \lim_{\epsilon \to 0} \frac{\ln(\ell/\epsilon)}{\ln(1/\epsilon)}$$

$$D_0 = \lim_{\epsilon \to 0} \frac{\ln \ell + \ln(1/\epsilon)}{\ln(1/\epsilon)} \to 1$$

$$D_0 = 1$$

c) Closed Curve – Area A



$$D_0 = \lim_{\epsilon \to 0} \frac{\ln A/\epsilon^2}{\ln(1/\epsilon)}$$

$$= \lim_{\epsilon \to 0} \frac{\ln A + 2\ln(1/\epsilon)}{\ln(1/\epsilon)}$$

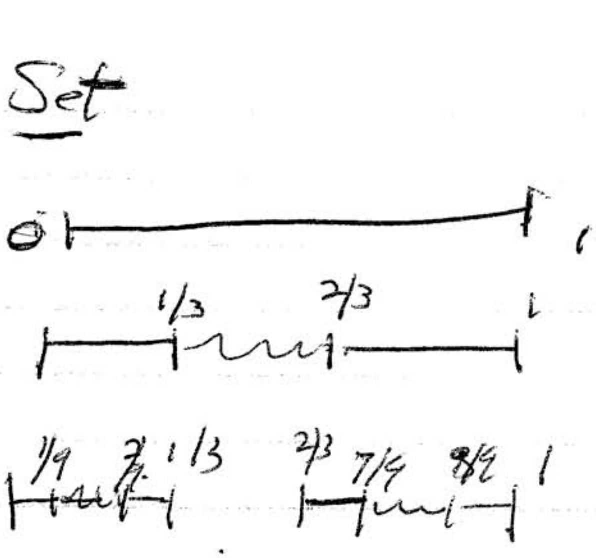
$$D = 2$$

Now something juicier:

~ Middle Third Cantor Set

Define odd set:

Chop out middle 1/3.



for each n, cover with $\overset{\tilde{N}}{2^n}$ pieces, $\underset{\epsilon}{(1/3)^n}$ length:

$$D_0 = \lim_{\substack{n\to\infty \\ (\epsilon\to 0)}} \frac{\ln 2^n}{\ln\left(1/(\frac{1}{3})^n\right)}$$

$$= \lim_{n\to\infty} \frac{n \ln 2}{n \ln 3}$$

$$\boxed{\begin{aligned} D_0 &= \ln 2/\ln 3 \\ D_0 &= .63\ldots \end{aligned}}$$

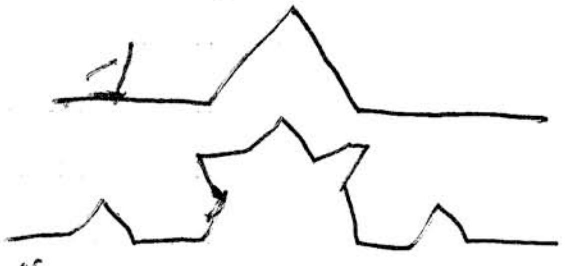
- fractal dimension
- $0 < D_0 < 1$
- embedded in $D = 1$ space $\quad D_0 < D_{embedd}$
- note $\Leftrightarrow$ D $\Leftrightarrow$ power law

i.e. $\tilde{N}(\epsilon) \sim \epsilon^{-D_0}$

box counting dimension

Fractals are self-similar.

~ Could also encounter Koch Curve

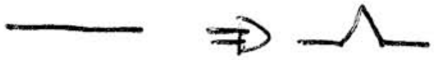


1 4

16

1/3

i.e. every



upon iteration

$$D_0 \equiv \lim_{\epsilon \to \infty} \frac{N(\epsilon)}{\ln(1/\epsilon)}$$

$$= \lim_{\substack{\epsilon \to \infty \\ n \to \infty}} \frac{4^n}{\ln\left[1/(1/3)^n\right]}$$

$$= \lim_{n \to \infty} \frac{2n \ln 2}{n \ln 3} = \frac{2 \ln 2}{\ln 3}$$

$$\boxed{\begin{array}{l} D_0 = 2\ln 2/\ln 3 \\ D_0 \sim 1.2\ldots \end{array}}$$

- here example of a fractal which thickens, i.e. $K D < 2$ · Need embedded in 2D.
- akin "coast-of-Britain" problem (Richardson '61, Mandelbrot ...)

i.e. increased resolution reveals
longer, more convoluted coastline.

$$\tilde{N}(\epsilon) \sim \epsilon^{-D_0}$$

$\Leftrightarrow$ rougher on smaller scale...
(N increases with $\epsilon \downarrow$).

Why Fractals?

→ self-similar structures with
dimension < dimension of
embedding space (i.e. 3)

⇒ natural candidates to describe:

- intermittent dissipation events
- geometry of dissipative
structures in intermittent
turbulence / cascades

→ cascade is due hierarchical, embedded
processes; dissipative structure
does not fill space

$D_0 \Leftrightarrow$
- intermittency correction to K41
spectrum!

→ The idea:

- Fractal structure is picture/phenomenology of observed departure from K41 spectrum.

- trends of scalings → plausible (i.e. fit)

but

- theory, based on N-S Eqn, does not "predict" D_0!

N.B. { Geometrically/symmetry motivated phenomenology is extremely useful i.e. Landau-Ginzburg, etc.

which brings us to:

→ <u>β - model</u> (Frisch, Sulem, Nelkin)

→ basic idea: (Mandelbrot)

- cascade active region is self-similar fractal structure, with $D < 3$.
- dissipation events are 'patchy'
- forces correction to K41.

→ <u>Analysis</u>

- why intermittency? ⇔ physics of cascade.

⇒ vortex stretching is <u>very</u> nonlinear

enthalpy

$$\partial_t \underline{v} + \underline{v}\cdot\nabla\underline{v} = -\nabla w + \nu\nabla^2\underline{v}$$

$$\partial_t \underline{v} = -\nabla\left(w + \frac{v^2}{2}\right) + \underline{v}\times\underline{\omega} + \nu\nabla^2\underline{v}$$

$$\underline{\omega} = \underline{\nabla}\times\underline{v}$$ → vorticity (key physics)

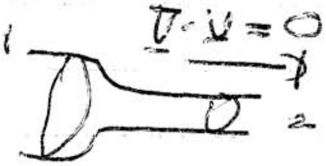
$\Rightarrow \underline{\nabla}\times \Rightarrow$

$$\boxed{\partial_t \underline{\omega} = \underline{\nabla}\times(\underline{v}\times\underline{\omega}) + \nu \nabla^2\underline{\omega}}$$

vorticity advection eqn.

$$\frac{d\underline{\omega}}{dt} = \underline{\omega}\cdot\underline{\nabla}\,\underline{v} + \nu\nabla^2\underline{\omega}$$

$\hookrightarrow$ vortex stretching



Kelvin Thm
$\oint \underline{v}\cdot d\underline{\ell} = \text{const.}$

$$\sim \omega^2 + \nu\nabla^2\underline{\omega}$$

heuristic only

$\Rightarrow$ fast (nearly explosive) growth of vorticity, enstrophy
$\langle\omega^2\rangle$ production to dissipn.

$\Rightarrow$ bursts, etc.

$\Rightarrow$ vortex stretching feeds on self $\Leftrightarrow$ localized process.

$\rightarrow$ so, "patchy" "embedded" cascade:

$\hookrightarrow$ occupation factor

$$\bar{\epsilon} \sim \beta_n \frac{V_n^3}{\ell_n}$$

$\bar{\epsilon}$: mean dissipn. rate

$\beta_n \equiv$ { fraction of space active on nth step cascade

N.B.

- $\oint \underline{v} \cdot d\underline{l} = \int \underline{\omega} \cdot d\underline{a} = \text{const}$

$$\omega_1 r_1^2 \sim \omega_2 r_2^2 \Rightarrow \text{vorticity increases on small scale}$$

- N.B. analogy

$$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = \eta \underline{J}$$

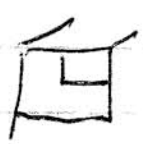
$$\underline{\nabla} \times \underline{E} = -\frac{1}{c} \frac{\partial \underline{B}}{\partial t}$$

$$\Rightarrow \partial_t \underline{B} = \underline{\nabla} \times \left(\frac{\underline{v} \times \underline{B}}{c} \right) + \eta \nabla^2 \underline{B}$$

β_n ? → fraction of volume active in n^{th} step of cascade

N.B. – if each eddy scale $\ell \to \ell/2$ per step

then # children to fill space per step is $2^3 = 8$

i.e. 

$$\Rightarrow \quad \beta \equiv N/2^3 < 1$$

N ← # off-spring

occupation reduction factor.

$$\beta_n = (\beta)^n = (N/2^3)^n \quad \to n \text{ steps.}$$

→ now, interpretation only

$$N \equiv 2^D \qquad D < 3$$

(simply an interpretation)

D → box counting dimension

$$\boxed{\beta_n = \left(2^{D-3}\right)^n}$$

So, taking mean energy balance

$$\bar{\epsilon} = \beta_n \frac{v_n^3}{\ell_n} \qquad \beta_n = 2^{n(D-3)} = \left(\frac{\ell_0}{\ell_n}\right)^{(D-3)}$$

$$= \left(\frac{\ell_n}{\ell_0}\right)^{3-D} \frac{v_n^3}{\ell_n}$$

$\Rightarrow$

$$v(\ell_n) \sim (\bar{\epsilon}\ell_n)^{1/3} \underbrace{(\ell_n/\ell_0)^{-1/3(3-D)}}_{\text{correction due } D \neq 0 \; \Rightarrow \text{ induces explicit } \ell_0}$$

fraction of active

$$E_n \sim \beta_n v(\ell_n)^2 \sim \underbrace{\bar{\epsilon}^{2/3} \ell_n^{2/3} (\ell_n/\ell_0)^{-2/3(3-D)}}_{\text{velocity in active region}} \underbrace{\left(\frac{\ell_n}{\ell_0}\right)^{(3-D)}}$$

$$\sim \bar{\epsilon}^{2/3} \ell_n^{2/3} (\ell_n/\ell_0)^{(3-D)/3}$$

and so

$$\boxed{\begin{aligned} E(\ell_n) &\sim \bar{\epsilon}^{2/3} \ell_n^{2/3} (\ell_n/\ell_0)^{(3-D)/3} \\ E(k) &\sim \bar{\epsilon}^{2/3} k^{-5/3} (k \ell_0)^{-\frac{1}{3}(3-D)} \end{aligned}}$$

→ correction to K41, in proportion $3-D$

→ slight <u>steepening</u> of spectrum

can deduce effective dimension from fit to spectral data.

Finally, dissipation scale changes;

<u>i.e.</u> $\frac{\nu}{\ell_d^2} = \frac{V(\ell_d)}{\ell_d}$ R_e [redacted] $\sim \frac{\ell_0 V_0}{\nu}$

[redacted]

<u>but</u>

$V(\ell_d) \sim \bar{\epsilon}^{1/3} \ell_d^{1/3} (\ell_d/\ell_0)^{-\frac{(3-D)}{3}}$ $\bar{\epsilon} \sim V_0^3/\ell_0$

$$\Rightarrow \boxed{l_d \sim l_0 \left(Re\right)^{-3/1+D}}$$

$$Re = \frac{l_0 V_0}{\nu} = \frac{\epsilon^{1/3} l_0^{4/3}}{\nu}$$

$D = 3$

$$l_d \sim l_0 \left(\frac{\epsilon^{1/3} l_0^{4/3}}{\nu}\right)^{-3/4}$$

$$\sim \cancel{\frac{l_0}{l_0}} \, \epsilon^{-1/4} \, \nu^{+3/4}$$

$$l_d \sim \nu^{3/4} / \epsilon^{1/4} \, , \quad D = 3.$$

modified for $D < 3$.